Math 255B Lecture 9 Notes

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1 Adjoints of Unbounded Operators

1.1 Adjoints

Last time, we showed that if $P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ on $L^2(\Omega)$ where $\Omega \subseteq \mathbb{R}^n$ is open and $a_\alpha \in C^\infty(\Omega)$, then we get a **minimal realization**: $P_{\min}$ with $D(P_{\min}) = \{u \in L^2 : \exists \varphi_n \in C_0^\infty(\Omega) : \varphi_n \to u, P\varphi_n \text{ conv.}\}$ given by $P_{\min} u = \lim_{n \to \infty} P\varphi_n$. We also defined the **maximal realization** $P_{\max}$ with $D(P_{\max}) = \{u \in L^2 : Pu \in L^2\}$, where Pu is taken in the sense of distributions. Here, we have $P_{\min} \subseteq P_{\max}$, where both of these are closed operators.

Recall the definition of an adjoint: In a Hilbert space H, if $T \in \mathcal{L}(H, H)$, the **adjoint** $T^* \in \mathcal{L}(H, H)$ is defined by $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x, y \in H$. For unbounded operators, we will define this, paying attention to the domains.

Definition 1.1. Let $T : D(T) \to H$ be densely defined. We define the **adjoint** T^* by

$$D(T^*) = \{v \in H : \exists f \in H \text{ s.t. } \langle Tu, v \rangle = \langle u, f \rangle \ \forall u \in D(T)\},$$

$$T^* v = f.$$

Remark 1.1. The requirement that T is densely defined is crucial to this definition. $D(T)$ is dense, so f is unique if it exists. In particular, $\langle Tu, v \rangle = \langle u, T^* v \rangle$ for all $u \in D(T)$ and $v \in D(T^*)$.

Remark 1.2. By the Riesz representation theorem,

$$D(T^*) = \{v \in H : \exists C = C_v > 0 \text{ s.t. } |\langle Tu, v \rangle| \leq C\|u\|, u \in D(T)\}.$$

1.2 Examples: adjoints of differential operators

Example 1.1. Let $\Omega \subseteq \mathbb{R}^n$ be open, $P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ with $a_\alpha \in C^\infty(\Omega)$, where $D = \frac{1}{i}\partial$. Let $P_\Omega = P$ with $D(P_\Omega) = C_0^\infty(\Omega)$. Let's compute P_Ω^*.

First, associated to P is the **formal adjoint** P^* defined by $\langle Pu, v\rangle_{L^2} = \langle u, P^*v\rangle_{L^2}$ for all $u, v \in C_0^\infty(\Omega)$ (such an operator exists for any differential operator). We can calculate the formula using integration by parts:

$$P^*v = \sum_{|\alpha|\le m} D_x^\alpha(\overline{a_\alpha(x)v}).$$

So P^* is a differential operator of order m with C^∞ coefficients.

To compute the adjoint P_Ω^*, we have

$$\begin{aligned} D(P_\Omega^*) &= \{v \in L^2 : \exists f \in L^2 \text{ s.t. } \langle Pu, v\rangle_{L^2} = \langle u, f\rangle \ \forall u \in C_0^\infty(\Omega)\} \\ &= \{v \in L^2 : P^*v = f \in L^2\}, \end{aligned}$$

where P^*v is taken in the sense of distributions. In other words, $D(P_\Omega^*) = \{v \in L^2 : P^*v \in L^2\} = D(P_{\max}^*)$, the maximal realization of the formal adjoint, and $P_\Omega^* v = P^*v$.

Sometimes, we can give a nice local description of the domain of the adjoint.

Example 1.2. Assume that $P = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$ is **elliptic** in the sense that if $p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ for $x \in \Omega, \xi \in \mathbb{R}^n$, then $p(x,\xi) \neq 0$ for all $x \in \Omega, \xi \neq 0$. Then we have

$$\{v \in L^2 : P^*v \in L^2\} \subseteq H^m_{\text{loc}}(\Omega) = \{u \in L^2_{\text{loc}}(\Omega) : \partial^\alpha u \in L^2_{\text{loc}}(\Omega)\ \forall |\alpha| \le m\},$$

a local Sobolev space.

1.3 The graph of the adjoint

Proposition 1.1. *Let $T : D(T) \to H$ be densely defined. The graph of the adjoint is*

$$G(T^*) = [V(\overline{G(T)})]^\perp,$$

where $V : H \times H \to H \times H$ sends $(u, v) \mapsto (v, -u)$.

Remark 1.3. Taking the closure is a matter of taste. Since we are taking the orthogonal complement, it does not matter whether or not we close the graph or not, since the result will be closed.

Proof. When $u \in D(T)$ and $(v, w^*) \in H \times H$, we have

$$\langle V(u, Tu), (v, w^*)\rangle_{H\times H} = \langle Tu, v\rangle - \langle u, w^*\rangle .$$

The right hand side is 0 for all $u \in D(T)$ if and only if $v \in D(T^*), T^*v = w^*$. This is equivalent to $(v, w^*) \in G(T^*)$. The left hand side is 0 for all $u \in D(T)$ iff $(v, w^*) \in [V(G(T))]$. So $G(T^*) = [V(G(T))]^\perp = [V(\overline{G(T)})]^\perp$. □

Corollary 1.1. *T^* is closed.*

Remark 1.4. If densely defined operators $T_1 \subseteq T_2$ in the sense that $G(T_1) \subseteq G(T_2)$, then $T_2^* \subseteq T_1^*$.

Is T^* densely defined?

Proposition 1.2. *T is closable if and only if $D(T^*)$ is dense. In this case, $(T^*)^* = \overline{T}$.*

Proof. ($\implies$): Assume there is a nonzero $w \in H$ such that $w \perp D(T^*)$. Then for every $v \in D(T^*)$,

$$\langle (0, w), (T^* v, -v) \rangle_{H \times H} = 0,$$

so $(0, w) \in [V(G(T^*))]^\perp = V(G(T^*)^\perp)$. Recall that $G(T^*) = [V(\overline{G(T)})]^\perp$, so $(0, w) \in V(V(\overline{G(T)}))$. $V^2 = -1$, so $(0, w) \in G(T)$. So $w = 0$, as T is closable.

($\impliedby$): The proof is a similar computation. $\square$